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ENERGY SOLUTION TO BUCKLING OF SPHERICAL SHELLS BY USE OF THE RITZ METHOD

by

G. E. Patrick, Jr.

Department of the Army Project No. 1A130001A039

Stress and Thermodynamics Branch
Structures and Mechanics Laboratory
Directorate of Research and Development
U. S. Army Missile Command
Redstone Arsenal, Alabama

ABSTRACT

A closed solution was ~~has been~~ developed to calculate the critical buckling load of a uniformly loaded sphere which yields results that are in close agreement with test. This procedure permits the calculation of buckling loads of spherical shells with varying shell segments, varying loading (provided axisymmetric and radial), and a varying thickness.

ACKNOWLEDGMENT

The author wishes to thank Mr. Troy Smith for his assistance toward the completion of this report.

SYMBOLS

R	radius
h	shell thickness
ϵ	modulus of elasticity
u	Poisson ratio
P	external load
ϵ_{ϕ}	strain, meridional
ϵ_{θ}	strain, circumferential
ϕ	angle, meridional
θ	angle, circumferential
Δ	denotes a change
T	external work
V	internal work
I	total energy
A, B	arbitrary constants
w	deflection, radial
v	deflection, circumferential
σ	critical stress (PR/2h)

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INTRODUCTION

Numerous articles have been written with the intent of trying to obtain a solution for the buckling of spherical shells. Some solutions have been found but are based upon small angle approximations and uniform load.

The purpose of this report is to try to obtain a solution without the use of small angle approximations or the necessity of uniform load and be applicable to any shell segment.

DISCUSSION

The problem under consideration is that of elastic buckling of a thin spherical shell subjected to a uniform load.

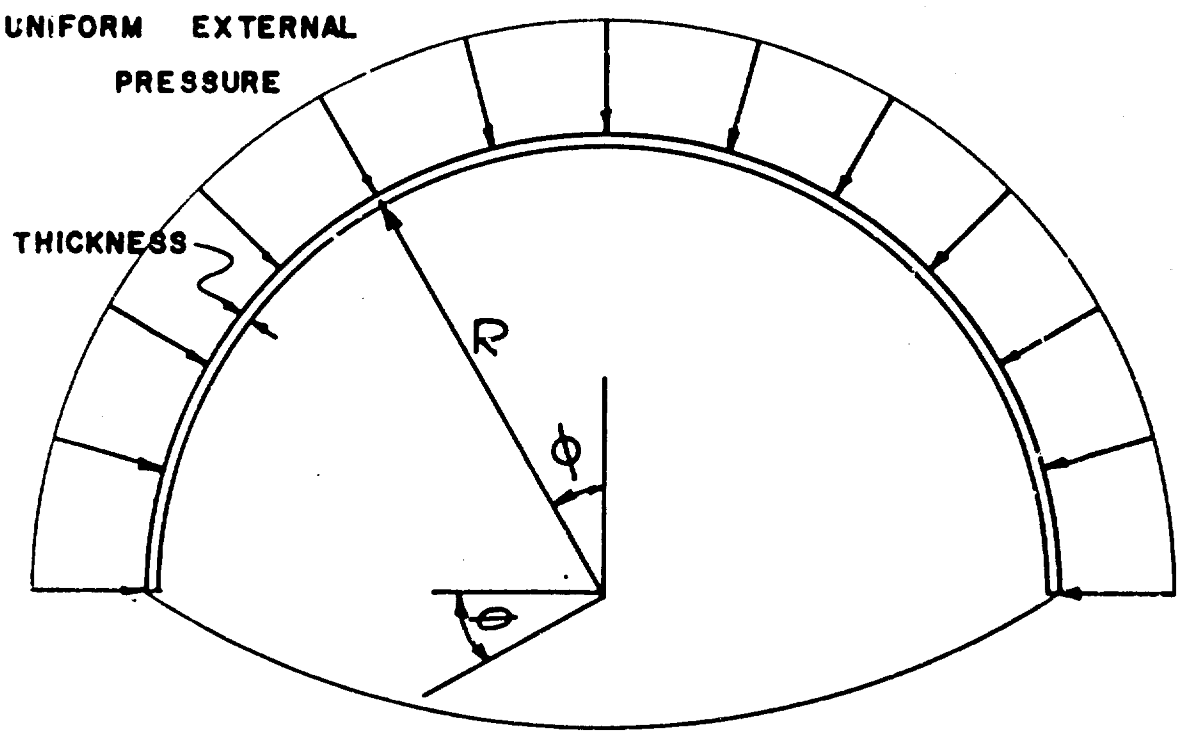

Figure 1

The principle of virtual work shall be employed. Several authors (Ref. 1, 2, and 3) give the form of the energy expressions. However, modifications are required to be acceptable for use here.

The total energy of a system consists of three parts:

1. Energy due to bending the membrane
2. Energy due to stretching the membrane
3. External work

Only the last of the three energy equations shall differ from those given in References 1, 2, and 3.

The energy that is contributed by bending shall be neglected as it is but a small percentage of the total amount in the shell. However, because of bending there is a change in surface area.

The stress distribution shall be assumed such that the transverse cross sections which were originally plane and normal to the centerline remain plane after bending. Let ab and cd be two adjacent cross sections of the shell (see Fig. 2) and $d\phi$ be the small angle between ab and cd before bending. As the load is applied, bending of the cross section occurs and section cd rotates with respect to ab about the neutral axis nn by a small angle $\Delta d\phi$. The angle is positive if the initial curvature of the shell is reduced. Because of the rotation the fibers on the outer surface are compressed and the fibers on the inner surface are put in tension. Let y be the distance from the centroidal axis perpendicular to the plane of bending, taken positive in the direction toward the center of curvature of the centerline of the element and e is the distance to the neutral axis nn from the centroidal axis. It can be seen that the extension of any fiber during bending is $(y - e)\Delta d\phi$ and that the unit elongation of the fiber is

$$\epsilon_\phi = \frac{(y - e)\Delta d\phi}{(R - y)d\phi}$$

The stress distribution is no longer linear but follows a hyperbolic law. Assuming that the neutral axis and the centroidal axis are in the same plane, that the greatest stress is produced in the innermost fibers, and that the thickness is small in comparison with the radius, we can write

$$\epsilon_\phi = -\frac{h\Delta d\phi}{2Rd\phi} \tag{1}$$

Note: The minus sign indicates inner surface.

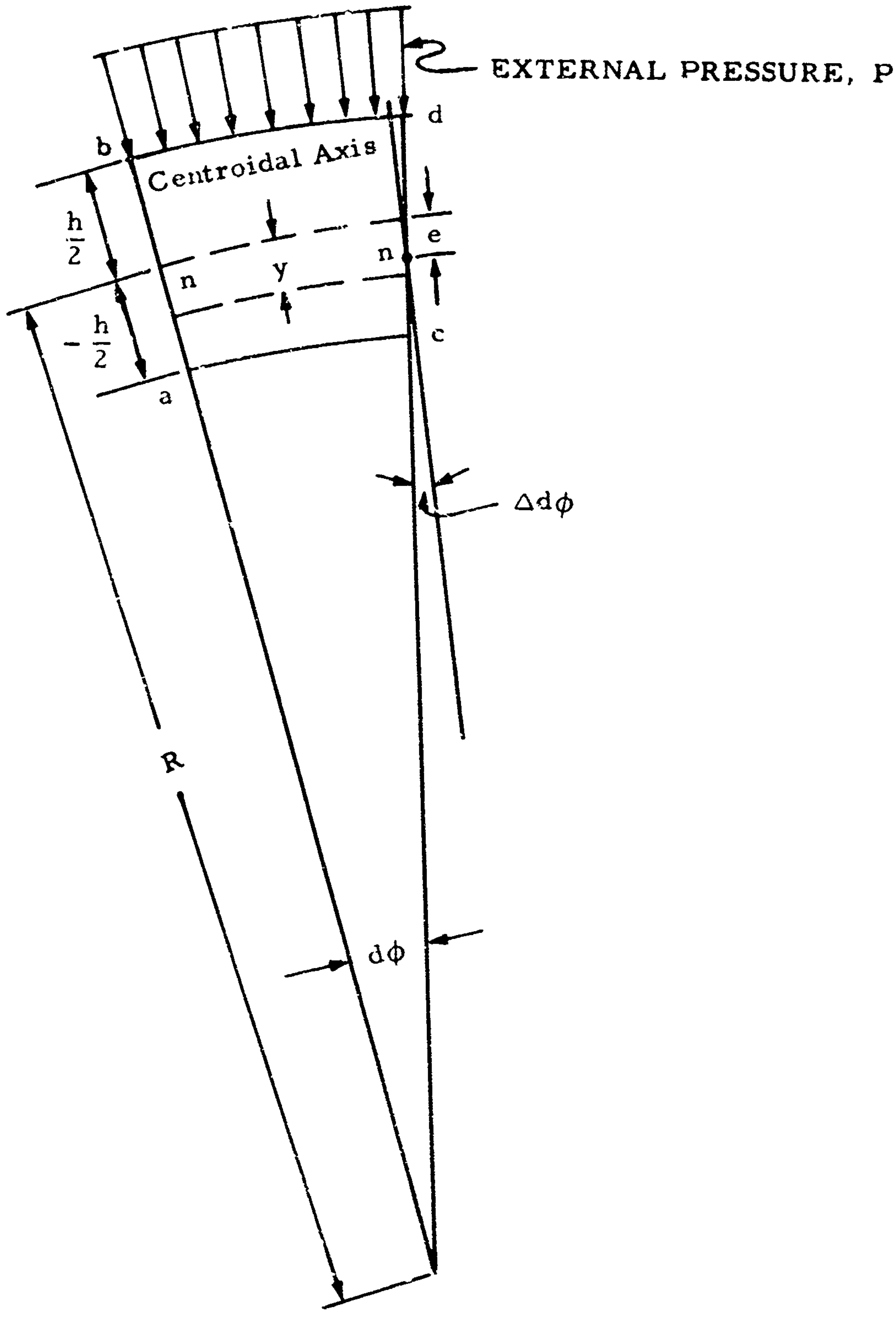
EXTERNAL PRESSURE, P
d
b
Centroidal Axis
$\frac{h}{2}$
e
n
n
y
$-\frac{h}{2}$
c
a
$\Delta d\phi$
R
$d\phi$

Figure 2

Similarly,

$$\epsilon_\theta = -\frac{h\Delta d\theta}{2Rd\theta} \tag{2}$$

Thus, the decrease in the angles during bending is

$$\Delta d\phi = -\frac{2\epsilon_\phi R a\phi}{h} \tag{1a}$$

$$\Delta d\theta = -\frac{2\epsilon_\theta R d\theta}{h} \tag{2a}$$

The original area over which the external load was applied is equal to

$$dA = Rd\phi a d\theta$$

where

$$a = R \sin \phi$$

By the bending of the membrane, this infinitesimal area will change to

$$dA' = R(1 + \Delta)d\phi(1 + \Delta)ad\theta$$

or by substituting equations 1a and 2a

$$dA' = R^2\left(1 - \frac{2\epsilon_\phi R}{h}\right)\left(1 - \frac{2\epsilon_\theta R}{h}\right) \sin \phi d\phi \tag{3}$$

Expanding equation 3 and neglecting small terms we get

$$dA' = R^2\left(1 - \frac{2\epsilon_\phi R}{h} - \frac{2\epsilon_\theta R}{h}\right) \sin \phi d\phi d\theta \tag{4}$$

The external work is given as

$$T = -\int_{Area} PwdA \tag{5}$$

Now substitute dA' for dA and we get

$$T = -R^2 \int_\phi \int_\theta Pw\left(1 - \frac{2\epsilon_\phi R}{h} - \frac{2\epsilon_\theta R}{h}\right) \sin \phi d\phi d\theta \tag{5a}$$

The internal work neglecting bending is

$$V = R^2 \int_\phi \int_\theta \left(\epsilon_\phi N_\phi + \epsilon_\theta N_\theta\right) \sin\phi d\phi d\theta$$

The magnitude of the compressive stresses in the shell we shall call

$$\overline{N}$$

If we assume that the quantities v and w represent components of small displacements during buckling from the compressed spherical shell form, then the N_ϕ and N_θ compressive forces differ but little from the compressive forces $\overline{N}$, thus

$$N_\phi = \overline{N}_\phi + N'_\phi$$

and

$$N_\theta = \overline{N}_\theta + N'_\theta$$

Since the external load P produced the $\overline{N}$ forces in the shell with the original deformation w, the external work shall include only the small increase as will the internal work.

Therefore, the internal work becomes

$$V = R^2 \int_\phi \int_\theta \left(N'_\phi \epsilon_\phi + N'_\theta \epsilon_\theta\right) \sin\phi d\phi d\theta \qquad (6)$$

However,

$$N'_\phi = \frac{Eh}{(1 - u^2)}\left(\epsilon_\phi + u\epsilon_\theta\right)$$

and

$$N'_\theta = \frac{Eh}{(1 - u^2)}\left(\epsilon_\theta + u\epsilon_\phi\right)$$

Substituting into equation 6

$$V = \frac{R^2Eh}{(1-u^2)}\int_\phi\int_\theta \left(\epsilon^2_\phi + \epsilon^2_\theta + 2u\epsilon_\phi\epsilon_\theta\right) \sin\phi d\phi d\theta \tag{6a}$$

Then equation 5a becomes

$$T = +\frac{2R^3}{h}P\int_\phi\int_\theta \left(w\epsilon_\phi + w\epsilon_\theta\right) \sin\phi d\phi d\theta \tag{5b}$$

The total energy of the system, $T + V$, is equal to a constant, I.

$$T + V = I \tag{7}$$

This total potential energy does not change when the structure passes from an equilibrium position to an infinitesimal, adjacent position.

Therefore, the structure will be in a stable position if the total potential energy I is a stationary value and is a minimum.

According to the mathematical rule, I will be a minimum if the second variation $\delta^2 I$ is positive for any virtual displacement.

The Ritz method is very general and applies to all problems in mechanics and physics which may be considered as problems of the calculus of variations.

If the deflection w is expressed in the form

$$w = a_1\phi_1\theta_1 + a_2\phi_2\theta_2 + \dots + a_n\phi_n\theta_n$$

where the total function satisfies the boundary conditions. The coefficients a are arbitrarily chosen parameters.

If this deflection curve is introduced into the energy equations, we arrive at an expression for the total potential energy I as a function of the n parameters a. If w is to be regarded as a solution to the extremum problem, it must satisfy the condition of extremum

$$T + V = I = \text{stationary}$$

The problem is then a maximum - minimum of calculus.

By taking the partial derivative of the total potential with respect to each of the arbitrary parameters and set equal to zero, we arrive at a system of n homogeneous linear equations from which the parameters a are to be determined. This system of equations does not have a solution different from zero unless the determinant D of its coefficients is equal to zero. Therefore,

$$D = 0$$

is the equation of degree n in the unknown force P and from this the stability condition from which P can be determined. The smallest root is the $P = \sigma$.

Therefore, the arbitrary parameters a remain arbitrary because the determinant D of their coefficient vanishes.

Success or failure in applying the Ritz method to any problem is content upon the individuality of the problem and the analysis may become unnecessarily lengthy and laborious if the series is not properly selected.

The Ritz method, although giving values higher than actual, offers the means for the approximate solution to buckling problems in those cases where the exact solution may become too difficult or is not practicable.

Timoshenko provided a concept similar to the Ritz method. However, whether Timoshenko or the Ritz method is used, results are higher than actual.

For certain buckling problems a good approximation can be obtained with only one term in the series.

An extension of the Ritz method, which was made by Trefftz in 1935, is not included in this report.

Substituting the linear strain - displacement relationships (Ref. 4)

$$\epsilon_\phi = \frac{1}{R}\left(\frac{dv}{d\phi} - w\right)$$

$$\epsilon_\theta = \frac{1}{R}\left(v \cot \phi - w\right)$$

into equations 6a and 5b we get the total potential I of the shell

$$I = \frac{Eh}{1-u^2} \int_\phi \int_\theta \left[\left(\frac{dv}{d\phi} \right)^2 + 2(1+u) \left(w^2 - w\frac{dv}{d\phi} - vw \cot \phi \right) \right.$$

$$\left. + v^2 \cot^2 \phi + 2uv\frac{dv}{d\phi} \cot \phi \right] \sin \phi d\phi d\theta$$

$$+ \frac{2PR^2}{h} \int_\phi \int_\theta \left(w\frac{dv}{d\phi} + vw \cot \phi - 2w^2 \right) \sin \phi d\phi d\theta \qquad (8)$$

By substituting a selected deflection for w and v we can obtain the following total potential

$$I = X_1A^2 + X_2AB + X_3B^2 \qquad (9)$$

where X is a function of (constant, P, E, h, R, and u) and A and B are arbitrary constants.

If we take the partial of equation 9 with respect to A then B, we get

$$\frac{\partial I}{\partial A} = 2X_1A + X_2B = 0$$

$$\frac{\partial I}{\partial B} = X_2A + 2X_3B = 0$$

Therefore, the determinant D is

$$D = 4X_1X_3 - X_2^2 = 0$$

From this we can solve for the critical value of the external pressure, P

For a complete sphere with a uniform load and $u = 0.3$, we obtain a critical stress as seen in Appendix A

$$\sigma = 0.282 \frac{Eh}{R}$$

which compares with the test values of

$$\sigma = 0.154 \frac{Eh}{R}$$

This calculated value is slightly higher than those given by test but the procedure allows for the calculation of any shell opening, loading (provided axisymmetric and radial) and the use of a varying thickness is expressed in terms of the meridional angle. The selected form of the deflection curve series must satisfy the boundary conditions of the specific problem.

Appendix A is a sample calculation for a two-term deflection curve for a hemisphere.

Figure 3 shows the buckling curve for varying shell openings from 10 to 90 degrees.

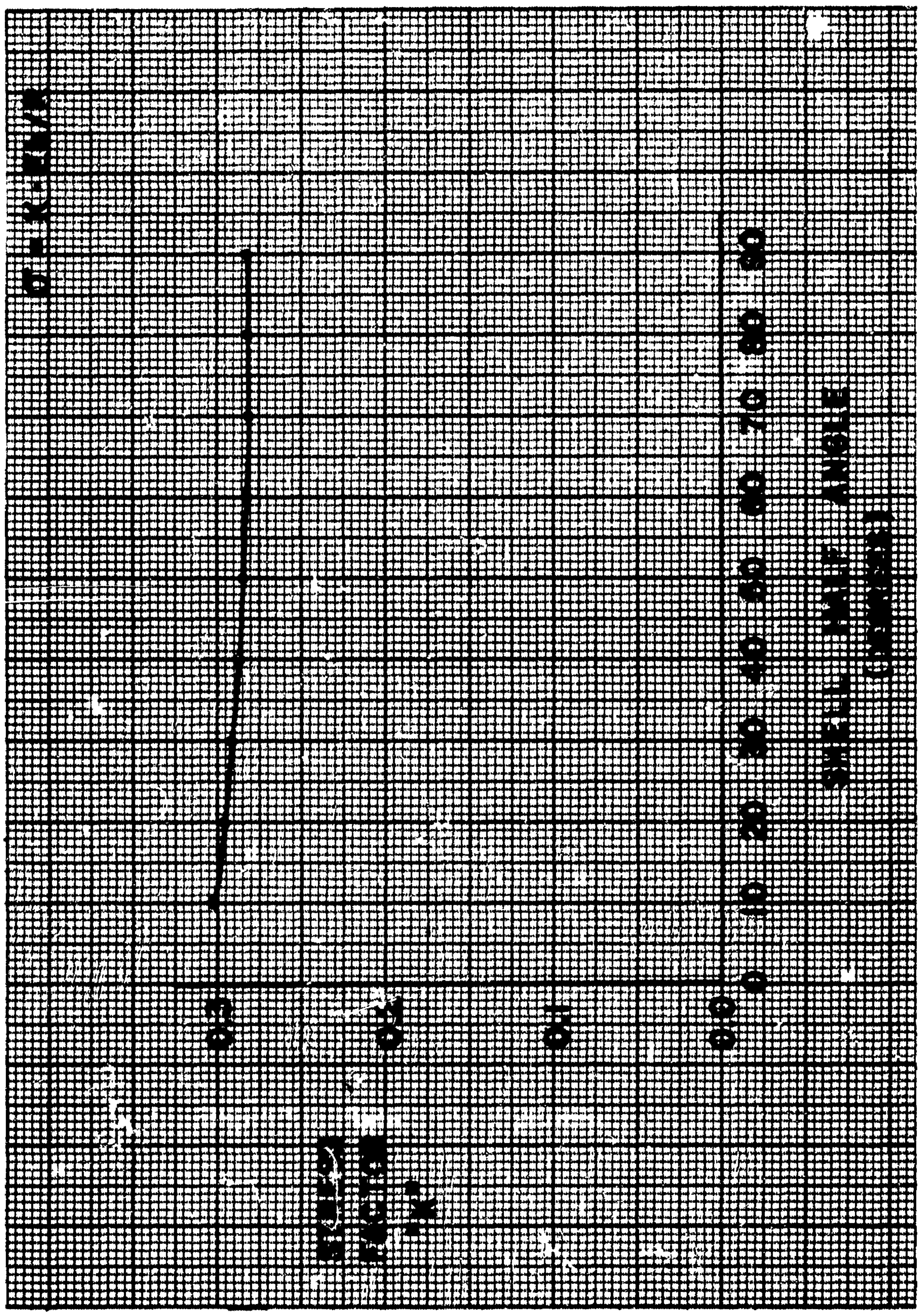

Figure 3

APPENDIX A
SAMPLE CALCULATIONS
HEMISPHERE
$\beta = 90°$

$$I = \frac{Eh}{1-u^2}\int_0^{2\pi}\int_0^{\beta}\left[\left(\frac{dv}{d\phi}\right) + 2(1+u)\left(w^2 - w\frac{dv}{d\phi} - vw\cot\phi\right)\right.$$

$$\left. + v^2\cot^2\phi + 2uv\frac{dv}{d\phi}\cot\phi\ \right] \sin\phi \, d\phi \, d\theta$$

Assume:

$$w = A_1\cos\phi\sin\theta + A_2\cos 3\phi\sin 3\theta$$

$$v = B_1\sin\phi\sin\theta + B_2\sin 3\phi\sin 3\theta$$

$$\frac{dv}{d\phi} = B_1\cos\phi\sin\theta + B_2\cos 3\phi\sin 3\theta \cdot 3$$

Substituting and noting

$$\int_0^{2\pi}\sin^2\left(\tfrac{1}{3}\right)\theta\, d\theta = \pi$$

$$\int_0^{2n}\sin\theta\sin 3\theta\, d\theta = 0$$

Performing the intergration, minimize the total energy with respect to A_1, A_2, B_1 & B_2 and set in determinant form, we get

APPENDIX A (Concluded)

Let $C_1 = \frac{Eh}{1-u^2}$ $\quad C_2 = \frac{2PR^2}{h}$

$1.733 C_1 - 1.333 C_2$	0.0	$-1.733 C_1 + 0.666 C_2$	0.0
0.0	$2.526 C_1 - 1.943 C_2$	0.0	$-4.011 C_1 + 1.543 C_2$
$-1.733 C_1 + 0.666 C_2$	0.0	$1.733 C_1$	0.0
0.0	$-4.011 C_1 + 1.543 C_2$	0.0	$11.088 C_1$

This gives a fourth order equation, the smallest positive root being

$$C_2 = 1.026 C_1$$

or the critical stress

$$\sigma = 0.282 \, Eh/R$$

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APPROVED:

JAMES F. GILMORE
Chief, Stress and Thermodynamics Branch

WILL A. LEWIS
Director, Structures and Mechanics Laboratory

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